Exam. Code : 211004
Subject Code : 4990

M.Sc. (Mathematics) 4th Semester

TOPOLOGY—II

Paper—MATH-582

Time Allowed—2 Hours] [Maximum Marks—100

Note :— There are ***eight*** questions of equal marks. Candidates are required to attempt any ***four*** questions.

1. (a) Prove that a topological space is completely normal if and only if for every pair A, B of subsets of X satisfying $\overline{A} \cap B = \phi$, $A \cap \overline{B} = \phi$ there exist disjoint open sets containing them.

   (b) Prove that a product of completely regular spaces is completely regular.

2. (a) Let $\mathbb{R}^{\omega}$ denotes the countable product of IR with itself in the product topology. Now prove the following :

   (i) $\mathbb{R}^{\omega}$ is Hausdörff

   (ii) $\mathbb{R}^{\omega}$ is normal.

   (b) Prove that every compact Hausdörff space is normal.

3. Prove Tychonöff Theorem, that is, an arbitrary product of compact topological spaces is compact.

4. (a) Let $f : X \to Y$ be a continuous bijection. If X is compact & Y is Hausdörff, show that f is homeomorphism.

(b) Prove that compactness implies limit point compactness but not conversely.

(c) Characterize all compact subspaces of IR in the finite complement topology.

5. (a) Let X be a completely regular space. Prove that there exists a compactification Y of X having the property that every bounded continuous function $f : X \to$ IR extends uniquely to a continuous map of Y into IR.

(b) Let X be a completely regular space and Y be a compactification of X satisfying the extension property of the statement in the preceding question 5(a). Given a continuous map $g : X \to C$ of X into a compact Hausdörff space C, prove that g extends uniquely to a continuous map $h : Y \to C$.

6. (a) Let X be a regular space with a countable basis. Show that there exists a countable collection of continuous functions $f_n : X \to [0,1]$ having the property that given any point $x_0 \in X$ and an open set $U \ni x_0$, there exists an index n such that $f_n(x_0) > 0$ and $f_n(X - U) = \{0\}$.

(b) Prove that a regular space with a countable basis is metrizable.

7. (a) Suppose that the nets $(x_\alpha)_{\alpha \in J} \to x$ in X and $(y_\alpha)_{\alpha \in J} \to y$ in Y. Prove that the net $(x_\alpha \times y_\alpha)_{\alpha \in J} \to x \times y$ in the product space $X \times Y$.

(b) Show that a simply ordered set under the simple order $<$ can be made into a directed set by the partial order $\leq$.

(c) Prove that a net $(x_\alpha)_{\alpha \in J}$ has a point x as an accumulation point if and only if some subnet of (x_α) converges to x.

8. (a) Let J be a directed set; $f : J \to X$ be a net in X. Let $f(\alpha) = x_\alpha$. if k is a directed set and $g : k \to J$ is a function s.t.

(i) $i \leq j \Rightarrow g(i) \leq g(j)$

(ii) $g(k)$ is cofinal in J,

prove that the composite functions $f \circ g : k \to X$ is convergent as a net if (x_α) is convergent.

(b) Define filter and ultrafilter. Describe a canonical way of converting nets into filters.

.